

A NOTE ON SURJECTIVE INVERSE SYSTEMS

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Abstract: Given an upward directed set I we consider surjective I -inverse systems $\{X_\alpha, f_{\alpha\beta} : X_\beta \longrightarrow X_\alpha | \alpha \leq \beta \in I\}$, namely those inverse systems that have all $f_{\alpha\beta}$ surjective. A number of properties of I -inverse systems have been investigated; such are the Mittag-Leffler condition, investigated by Grothendieck and flabby and semi-flabby I -inverse systems studied by Jensen. We note that flabby implies semi-flabby implies surjective implies Mittag-Leffler. Some of the results about surjective inverse systems have been known for some time. The aim of this note is to give a series of equivalent statements and implications involving surjective inverse systems and the systems satisfying the Mittag-Leffler condition, together with improvements of established results, as well as their relationships with the already known, but scattered facts. The most prominent results relate cardinalities of the index sets with right exactness of the inverse limit functor and the non-vanishing of the inverse limit – connections related to cohomological dimensions.

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For an (abelian) category \mathcal{C} with infinite products, and an (upward directed) ordered index set I , an I -inverse system $\{X_i, f_{ij}\}$ in \mathcal{C} is said to satisfy the *Mittag-Leffler condition* (or the *ML condition*, for short), if $\forall i \in I \exists j \geq i$ such that, $\forall k \geq j$ $f_{ik}X_k = f_{ij}X_j$, in other words, if for every $i \in I$, the decreasing sequence $\{f_{ik}X_k\}_{k \in I}$ of submodules of X_i stabilizes from certain index on; we will also say that $\{X_i, f_{ij}\}$ is *eventually stable*. Note that for any $i \in I$, $\{f_{ij}X_j\}_{j \geq i}$ is a decreasing family of subobjects of X_i . Then the *object of universal images* $X'_i = \inf_{j \geq i} f_{ij}X_j (= \cap_{j \geq i} f_{ij}X_j)$ of X_i exists and, for $f_i : X = \varprojlim X_i \longrightarrow X_i$ – the canonical morphism, $f_iX \subseteq X'_i$ with $f_{ij}X'_j \subseteq X'_i$, for $j \geq i$. This makes $\{X'_i, f_{ij}|X'_j = f'_{ij}\}$ an I -inverse system (the *inverse system of universal images*), with $\varprojlim X'_i = X$. If ML holds, for $\{X_i, f_{ij}\}$ then the restriction $f'_{ij} = f_{ij}|X'_j$ is surjective, for all $j \geq i$ (there is a $k \geq j$ such that, for every $l \geq k \geq j$, $f_{jl}X_l = f_{jk}X_k$, hence $X'_j = f_{jk}X_k$ and $f_{il}X_l = f_{ik}X_k$, thus we have $X'_i = f_{ik}X_k = f_{ij}X'_j$).

We now construct a useful example of a surjective I -inverse system $\{E_\alpha, \epsilon_{\alpha\beta} : \alpha \leq \beta \in I\}$ of non-empty sets, for every non-empty upward directed set I . The case when I has a maximal (the maximum) element is usually favorable in considerations about \varprojlim , hence we treat a more difficult case when I has no maximal elements.

For $\alpha \in I$ let E_α denote the set of ordered even-tuplets $(\alpha_1, \alpha_2, \dots, \alpha_{2n-1}, \alpha_{2n})$ of elements of I , with the following properties:

- (1) $\alpha_{2n-1} = \alpha$,
- (2) $\forall 1 \leq i \leq n, \alpha_{2i-1} \leq \alpha_{2i}$, and
- (3) $\forall 1 \leq j < i \leq n, \alpha_{2i-1} \not\leq \alpha_{2j-1}$.

All E_α are non-empty, by non-maximality; they are also disjoint. To construct the maps, let $\alpha \leq \beta$ and $(\beta_1, \dots, \beta_{2m-1}, \beta_{2m}) \in E_\beta$, let j be the smallest numeral such that $\alpha \leq \beta_{2j-1}$; define $\epsilon_{\alpha\beta} : E_\beta \longrightarrow E_\alpha$ by $\epsilon_{\alpha\beta}(\beta_1, \dots, \beta_{2m-1}, \beta_{2m}) = (\beta_1, \dots, \beta_{2j-2}, \alpha, \beta_{2j})$. To show that $\{E_\alpha, \epsilon_{\alpha\beta}\}$ is a surjective inverse system, start with an $x = (\alpha_1, \dots, \alpha_{2n}) \in E_\alpha$ (hence $\alpha_{2n-1} = \alpha$). Pick any $\gamma > \beta \geq \alpha$; we prove that $\epsilon_{\alpha\beta}(y) = x$, where $y = (\alpha_1, \dots, \alpha_{2n}, \beta, \gamma)$. The non-trivial point to prove is that $y \in E_\beta$: $x \in E_\alpha$ satisfies conditions (1)–(3), thus we need only prove that $\beta \not\leq \alpha_{2l+1}$, for $l = 1, 2, \dots, n-1$. This must be the case, for if $\beta \leq \alpha_{2l+1}$, then $\alpha < \beta$ would imply $\alpha_{2n-1} = \alpha < \alpha_{2l+1}$, but this is pre-

vented by condition (3) for x . The verification that $\epsilon_{\alpha\beta}(y) = x$ is straightforward.

G-sets: Given a group G , a non-empty set X is called a *G-set*, if G operates on X , i.e. if there is an operation $G \times X \longrightarrow X$, $(g, x) \mapsto gx$, such that $(g_1g_2)x = g_1(g_2x)$, and $ex = x$; a consequence is that if $gx = y$, then $x = g^{-1}y$. A non-empty G -set is *transitive*, if $\forall x_1, x_2 \in X$, there exists a $g \in G$ with $gx_1 = x_2$; if another element g_1 satisfies the same equation (i.e. $g_1x_1 = gx_1$), then $g^{-1}g_1$ belongs to the *isotropy subgroup (stabilizer)* $H = \{g \in G : gx_1 = x_1\}$ of x_1 . If G is the trivial group, then every set is trivially a G -set, but non-transitive in general. Every group G is a transitive G -set.

Theorem 1. *For a non-empty upward directed set I , the following are equivalent:*

- (1) *I has a maximal element, or it contains a countable cofinal sequence.*
- (2) *Every surjective I -inverse system $\{X_\alpha : f_{\alpha\beta}\}$ of non-empty sets has a non-empty inverse limit.*
- (3) *For every surjective map $\mathbf{g} = (g_\alpha)_{\alpha \in I} : \{E_\alpha : \epsilon_{\alpha\beta}\} \longrightarrow \{S_\alpha : \sigma_{\alpha\beta}\}$, of I -inverse systems of sets, such that all $\epsilon_{\alpha\beta}$ are surjective and $\sigma_{\alpha\beta}$ are injective, the induced inverse limit map $\varprojlim \mathbf{g} : \varprojlim \mathbf{E} \longrightarrow \varprojlim \mathbf{S}$ is likewise surjective.*
- (4) *Every I -inverse system of non-empty sets $\{X_\alpha : f_{\alpha\beta}\}$ that satisfies the ML condition has a non-empty inverse limit.*
- (5) *For every group G , every surjective I -inverse system of non-empty transitive G -sets has a non-empty inverse limit.*

Proof. (1) \Rightarrow (2): If there is a maximum $\alpha_0 \in I$, then pick an $x_{\alpha_0} \in X_{\alpha_0}$ and define $x_\alpha = f_{\alpha\alpha_0}(x_{\alpha_0})$, for every $\alpha \leq \alpha_0$; then $(x_\alpha)_{\alpha \in I} \in \varprojlim X_\alpha \neq \emptyset$ (notice that surjectivity of the inverse system is not needed here). If $J = \{\alpha_n : n \in \mathbb{N}\}$ is a countable cofinal subset of I , then $\varprojlim_I X_\alpha \cong \varprojlim_J X_\alpha$. Pick an $x_{\alpha_1} \in X_{\alpha_1}$; by surjectivity there is an $x_{\alpha_2} \in X_{\alpha_2}$ with $f_{\alpha_1\alpha_2}(x_{\alpha_2}) = x_{\alpha_1}$. Thus we construct inductively $(x_\alpha)_{\alpha \in J} \in \varprojlim_J X_\alpha \neq \emptyset$.

(2) \Rightarrow (3): Let $(s_\alpha)_{\alpha \in I} \in \varprojlim S_\alpha$; this means that $\forall \alpha \leq \beta, \sigma_{\alpha\beta}(s_\beta) = s_\alpha$. Define $E'_\alpha = g_\alpha^{-1}(s_\alpha) \neq \emptyset$. We have $\epsilon_{\alpha\beta}(E'_\beta) \subseteq E'_\alpha$. This is because $g_\alpha(\epsilon_{\alpha\beta}g_\beta^{-1}(s_\beta)) = \sigma_{\alpha\beta}g_\beta g_\beta^{-1}s_\beta = \sigma_{\alpha\beta}s_\beta = s_\alpha$. Hence, for $\epsilon'_{\alpha\beta} = \epsilon_{\alpha\beta}|_{E'_\alpha}$, we have an I -inverse system $\{E'_\alpha, \epsilon'_{\alpha\beta}\}$ of non-empty sets. Moreover, it is a surjective system, for if $y \in E'_\alpha$, surjectivity of $\epsilon_{\alpha\beta}$ ensures existence of an $x \in E'_\beta$ with $\epsilon_{\alpha\beta}(x) = y$.

This x is in E'_β , since $s_\alpha = \sigma_{\alpha\beta}s_\beta$ and $s_\alpha = g_\alpha\epsilon_{\alpha\beta}(x) = \sigma_{\alpha\beta}g_\beta(x)$, hence $\sigma_{\alpha\beta}s_\beta = \sigma_{\alpha\beta}g_\beta(x)$ and injectivity of $\sigma_{\alpha\beta}$ ensures $s_\beta = g_\beta(x)$.

(3) \Rightarrow (1): If I has a maximum, there is nothing to prove. Otherwise, consider a map $\mathbf{g} : \mathbf{E} \longrightarrow \mathbf{S}$ between the special I -inverse systems $\mathbf{E} = \{E_\alpha, f_{\alpha\beta}\}$ constructed in the introduction and $\mathbf{S} = \{S_\alpha, \sigma_{\alpha\beta}\}$, $S_\alpha = \{\alpha\}$, $\sigma_{\alpha\beta}(\beta) = \alpha$. Both of the systems are surjective and $\sigma_{\alpha\beta}$ are bijections. Define $g_\alpha(\alpha_1, \alpha_2, \dots, \alpha, \alpha_{2n}) = \alpha$; this \mathbf{g} is clearly surjective, and by the assumption, $\varprojlim \mathbf{g} : \varprojlim \mathbf{E} \longrightarrow \varprojlim \mathbf{S} = (\alpha)_{\alpha \in I}$ is also surjective, thus there is an $(e_\alpha)_{\alpha \in I} \in \varprojlim E$ that maps to $(\alpha)_{\alpha \in I}$. These e_α 's will produce a desired, cofinal sequence in I as follows: Looking into the set of ending coordinates of all the e_α 's, we see that that set is cofinal in I since, for every $\alpha \in I$, $\alpha = \alpha_{2n-1} < \alpha_{2n}$. This means that we would prove the claim if we show that this set either has a maximal element, or forms a countable sequence. We are assuming that there is no maximal element. Note that if e_α and e_β are tuplets of the same length $2n$, then $\alpha = \beta$, since if γ is chosen so that $\gamma > \alpha, \beta$, then $f_{\alpha\gamma}e_\gamma = e_\alpha$ and $f_{\beta\gamma}e_\gamma = e_\beta$; then the definition of the inverse system morphisms f_{ij} implies that the ending coordinates of e_α and e_β are certain coordinates γ_{2l} and γ_{2m} of e_γ . By the assumption of same length, $l = n = m$ and the ending coordinate of e_α is $\gamma_{2m} = \gamma_{2l} =$ the ending coordinate of e_β . The sizes of all the tuplets e_α are not bounded, for otherwise their ending coordinates would form a finite cofinal subset of I , hence it would have a maximal element; since the lengths are not bounded, consider the countable sequence of ending coordinates of each even-tuplet as the desired cofinal sequence.

(4) \Rightarrow (2) holds since surjective inverse systems are special cases of the ML systems.

For (2) \Rightarrow (5): Use the obvious forgetful functor.

(2) \Rightarrow (4): We have already mentioned, that the I -inverse system of universal images $f'_{\alpha\beta} : X'_\beta \longrightarrow X'_\alpha$ (where $X'_\beta = f_{\beta, \beta+1}(X_{\beta+1})$ – we can assume that $I = \mathbb{N}$, without loss of generality) is a surjective inverse system with $\varprojlim X'_\alpha = \varprojlim X_\alpha$, provided $\{X_\alpha, f_{\alpha\beta}\}$ satisfies the ML condition; since X_α 's are non-empty, then also X'_α 's are non-empty. By (2) $\varprojlim X'_\alpha \neq \emptyset$ and the claim is established.

(5) \Rightarrow (1): Let G be the (additively written) free abelian group on a set of generators g_{ij} ($i \leq j$ in I), and for each $\alpha \in I$ let H_α be the subgroup of G generated by the elements

$$(a) \quad g_{ij} + g_{jk} - g_{ik}, \quad k > j > i \geq \alpha.$$

Let us define X_α to be a transitive G -set with generator denoted by x_α and the stabilizer H_α , and, for $j \geq i$ define a morphism $X_j \rightarrow X_i$ by sending x_j to $g_{ij}x_i$. These clearly give a surjective I -inverse system of non-empty transitive G -sets. By the assumption its inverse limit is non-empty.

Suppose $y = (c_i x_i)_{i \in I} \in \varprojlim X_i$, for some $c_i \in G$. By definitions of \varprojlim and the maps $X_j \rightarrow X_i$ we have $g_{ij}(c_j x_i) = c_i x_i$ and, via the isotropy groups H_i , this translates into

$$(b) \quad g_{ij} + c_j - c_i \in H_i, \quad i \leq j.$$

Note again that all the generators of G occurring in (a) have both subscripts $\geq \alpha$, i.e. H_α is contained in the subgroup of G spanned by the generators with this property. It thus follows from (b) that c_i and c_j may differ (in the expansion of G) only in the terms with both subscripts in the set $\{i, j, k\}$ of the respective generators of G , $\geq i$. Hence for every $i < j$ and every g_{ij} , there is a $k_0 > i$ (say $k_0 = j$), such that, $\forall k \geq k_0$, all c_k contain in their expansion the same coefficient of g_{ij} ; in particular, this coefficient will be in all c_k with $k > j$.

If, contrary to our claim, I were of uncountable cofinality, then this coefficient cannot be nonzero for infinitely (countably) many g_{ij} , since we could find some k such that c_k involves infinitely many summands in the direct sum representation, which is impossible. Hence only finitely many different g_{ij} have nonzero eventual coefficient, hence we can form an element c of G in which each g_{ij} has this coefficient. "Translating" our element y of $\varprojlim X_i$ by c , and redefining the c_i in terms of this new y , we are reduced to the situation where the eventual coefficient of each g_{ij} is 0. Hence, by our earlier observations,

$$(c) \quad c_i \text{ involves no } g_{\alpha j} \text{ with } \alpha \leq i.$$

To complete our proof, let us now map G homomorphically into the free abelian group on generators $\{f_i : i \in I\}$ by the homomorphism D defined by

$$(d) \quad D(g_{ij}) = f_i - f_j.$$

Note that $\text{Ker}(D)$ contains all the subgroups H_i . Hence (b) and (c) give, respectively:

$$(e) \quad D(c_i) - D(c_j) = f_i - f_j \quad (i \leq j),$$

$$(f) \quad D(c_i) \text{ involves no } f_\alpha \text{ with } \alpha \leq i.$$

Looking at (e) in the light of (f) we see that the only f with subscript $\leq j$ which $D(c_i)$ involves is f_i , and this has coefficient 1. Hence fixing i and taking arbitrarily large j in this statement,

we conclude that f_i is the only f whatsoever that $D(c_i)$ involves with nonzero coefficient. Hence

$$(g) \quad D(c_i) = f_i.$$

But from (d) we see that D carries G into the subgroup of F in which the coefficients of the f 's sum to 0. This contradicts (g), thus the assumption of uncountability of I is false and we conclude that I either has a maximum element or is of countable cofinality. \square

Notes 1. [Bourbaki, 1961, §3, Th. 1] attributes to Mittag-Leffler an implication of the kind $(1) \Rightarrow (2)$ where spaces X_α were taken to be complete metrizable uniform spaces and the I -inverse systems in (2) satisfied the Mittag-Leffler condition, instead of surjective I -inverse systems. The proof $(3) \Rightarrow (1)$ and the construction of E_α 's is essentially that of [Henkin, 1950] (see also [Bourbaki, 1956, §1, Exercise 31]). The result $(5) \Rightarrow (1)$ and its proof is by [Bergman, 1998].

Given an (upward directed) index set I , the *surjective cohomological dimension* of I ($\text{scd}I$) is the largest natural number n with $\varprojlim^{(n)} A_i \neq 0$, for some surjective I -inverse system $\{A_i : i \in I\}$ of sets (or R -modules); if no such an n exists it is pronounced to be ∞ . Here $\varprojlim^{(0)} A_i$ is thought of as $\varprojlim A_i$. Since we assume to be working within categories with (infinite) direct products (that can be seen as inverse limits), we get then that $\text{scd}I \geq 0$, for all non-zero I .

Theorem 2. *For a non-empty upward directed set I , any of the equivalent statements (1)–(5) in Theorem 1 implies every of the following statements:*

- (6) *For every surjective I -inverse system of non-trivial free abelian groups (modules), its inverse limit is likewise non-trivial.*
- (7) *For every surjective I -inverse system $\{M_\alpha, f_{\alpha\beta} : \alpha, \beta \in I\}$ of non-trivial abelian groups (modules), its inverse limit is likewise non-trivial.*
- (8) *For every surjective I -inverse system \mathbf{A} of abelian groups (modules) and every $n \geq 1$, $\varprojlim^{(n)} A_i = 0$.*
- (9) *For every surjective I -inverse system \mathbf{A} of abelian groups (modules) and every exact sequence $0 \longrightarrow \mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{C} \longrightarrow 0$ of I -inverse systems, the corresponding sequence*

of inverse limits $0 \longrightarrow \varprojlim \mathbf{A} \longrightarrow \varprojlim \mathbf{B} \longrightarrow \varprojlim \mathbf{C} \longrightarrow 0$ is likewise exact.

(10) $\text{scd}I = 0$.

Proof. Note first that (8) \iff (9) \iff (10), since $\text{scd}I \geq 0$. For (8) \iff (9) use the exact sequence $0 \longrightarrow \varprojlim A_i \longrightarrow \varprojlim B_i \longrightarrow \varprojlim C_i \longrightarrow \varprojlim^{(1)} A_i \longrightarrow \dots$

(7) \Rightarrow (6): since the former is a more general than the latter.

(1) \Rightarrow (7): The proof of this is same, mutatis mutandis, as (1) \Rightarrow (2); we pick the starting element $x_{\alpha_0} \in M_{\alpha_0}$ (or $x_{\alpha_1} \in M_{\alpha_1}$) to be non-zero, to ensure that the resulting element in the limit is also non-zero.

(2) \Rightarrow (9): Let $\mathbf{A} = \{A_j, f_{ij}\}$, $\mathbf{B} = \{B_j, g_{ij}\}$, $\mathbf{C} = \{C_j, h_{ij}\}$, and exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_j & \xrightarrow{u_j} & B_j & \xrightarrow{v_j} & C_j \longrightarrow 0 \\
 (*) & & f_{ij} \downarrow & & g_{ij} \downarrow & & h_{ij} \downarrow \\
 0 & \longrightarrow & A_i & \xrightarrow{u_i} & B_i & \xrightarrow{v_i} & C_i \longrightarrow 0
 \end{array}$$

Denote also $v = \varprojlim v_i$, $u = \varprojlim u_i$. Given $0 \neq c = (c_i)_{i \in I} \in \varprojlim C_i$ (thus $h_{ij}(c_j) = c_i$) we need a $b = (b_i)_{i \in I} \in \varprojlim B_i$ with $v(b) = c$, i.e. $\forall j \ v_j(b_j) = c_j$. Denote $E_j = v_i^{-1}(c_j) \neq \emptyset$; it is non-empty, since the v_j 's are surjective. Denote now $e_{ij} = g_{ij}|_{E_j}$; it is straightforward to show that $\mathbf{E} = \{E_j, e_{ij}\}$ is an I -inverse system. We now show that it is surjective: To this end, start with a $b'_i \in E_i$, i.e. such that $v_i(b'_i) = 0$. By exactness in (*), there is an $a_i \in A_i$ with $u_i(a_i) = b'_i$ (**). By surjectivity of the f 's, there is an $a_j \in A_j$ with $f_{ij}(a_j) = a_i$ (***). Since $u_j(a_j) \in \text{Im } u_j = \text{Ker } v_j$, we have $v_j(u_j(a_j)) = 0$, i.e. $u_j(a_j) \in E_j$. Appeal again to (*), then (***) and (**) to get $g_{ij}u_j(a_j) = u_i f_{ij}(a_j) = u_i(a_i) = b'_i$; this proves surjectivity of all e_{ij} . By the assumption, $\varprojlim E_j \neq \emptyset$, hence for any b in that set we have $v(b) = c$ by the very construction, which proves the claim. \square

Notes 2. By way of universal images, an additional set of (equivalent) statements may be added with the word “surjective” replaced by the word the ML condition, in Theorem 2. (1) \Rightarrow (9) was proved in [Grothendieck, 1961] where \mathbf{A} is required to satisfy the Mittag-Leffler condition; [Goblot, 1970] replaces countability of the index set I by the requirement that I is well-ordered and that the participating objects and maps form a continuous

(smooth) system. [Jensen, 1972] replaces countability of I by a requirement that \mathbf{A} is semi-flabby. In an ongoing work we will show that $scdI = n$ if and only if $|I| = \aleph_n$, otherwise the surjective cohomological dimension is infinite ([Mitchell, 1973] shows similar result for the cohomological dimension: For a directed I , if $cfI = \aleph_k$, $-1 \leq k \leq \infty$, then $cdI = n + 1$ iff $k = n$). This will then establish equivalence of conditions in Theorem 1 with conditions in Theorem 2.

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